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# Yangian symmetry in the nonlinear Schrödinger hierarchy

M Mintchev†, E Ragoucy‡, P Sorba‡ and Ph Zaugg§

† INFN, Dipartimento di Fisica dell'Univ. di Pisa, Piazza Torricelli 2, 56100 Pisa, Italy
‡ LAPTH, Chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France
§ CRM, University de Montréal, CP 6128 Succ. Centre-ville, Montréal (Qc), Canada H3C 3J7

E-mail: `mintchev@difi.unipi.it`, `ragoucy@lapp.in2p3.fr`, `sorba@lapp.in2p3.fr` and `zaugg@CRM.UMontreal.CA`

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**Abstract.** We study the Yangian symmetry of the multicomponent quantum nonlinear Schrödinger hierarchy in the framework of the quantum inverse scattering method. We give an explicit realization of the Yangian generators in terms of the deformed oscillators algebra which naturally occurs in this framework.

## 1. Introduction

An increasing number of integrable systems with internal degrees of freedom have been shown to exhibit a Yangian symmetry. One of the earliest examples of this is perhaps the Haldane–Shastry spin chain [1] and the spin generalization of the Calogero–Sutherland model investigated in [2].

In the realm of integrable systems the quantum nonlinear Schrödinger (NLS) model distinguishes itself by being one of the most studied system and its simplest version played an important role in the development of the quantum inverse scattering method (QISM). Recently, the authors of [3] considered the quantum NLS model with spin $\frac{1}{2}$ fermions and repulsive interaction on the line and have unravelled the presence of a Yangian symmetry $Y(sl(2))$.

In this paper we consider the most general case of the quantum NLS model with $N$-component bosons or fermions and prove that the Yangian symmetry is $Y(sl(N))$. In addition we provide an explicit realization of the Yangian generators using the algebra of creation and annihilation operators for scattering states that is an essential part of the QISM, also known as the Zamolodchikov–Faddeev (ZF) algebra [4]. This approach makes it clear that the Yangian is actually a symmetry of the whole quantum integrable hierarchy whose lowest instance is the NLS model. As a by-product, we also obtain all the solutions to the equations of motion of the quantum NLS hierarchy.

The structure of the paper is as follows. In section 2 we summarize the QISM applied to the NLS model. Section 3 in devoted to the study of the Yangian symmetry of the NLS model. In section 4 we outline the construction of the Yangian generators, as well as the higher Hamiltonians of the NLS hierarchy, in terms of creation and annihilation operators of the ZF algebra, with the technical details being gathered in the appendices. Next we present a connection between the action of the Yangian algebra restricted to the subspaces of fixed particle number and a class of finite $W$-algebras. We conclude with possible extensions of this work.

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## 2. The multicomponent NLS model

We consider the $N$-components quantum NLS model with bosons or fermions and repulsive coupling. We first collect a few known results from the QISM applied to the model, most of which can be found in [5, 6] with more details.

The Hamiltonian of the NLS model is

$$H = \int \mathrm{d}x \left( \frac{\partial \phi^{\dagger i}}{\partial x} \frac{\partial \phi_i}{\partial x} - \rho g \phi^{\dagger i} \phi^{\dagger j} \phi_i \phi_j \right) \tag{2.1}$$

where the fields operators satisfy the equal time canonical commutation relations

$$[\phi_i(x), \phi^{\dagger j}(y)]_\rho = \delta_i^j \delta(x-y) \qquad [\phi_i(x), \phi_j(y)]_\rho = [\phi^{\dagger i}(x), \phi^{\dagger j}(y)]_\rho = 0.$$

The conventions are that $\rho = -1$ for bosons, $\rho = 1$ for fermions and $[\ ,\ ]_\rho$ stands, respectively, for the commutator or anti-commutator. Latin indices run from 1 to $N$, whereas Greek indices run from 1 to $N+1$. Repeated indices are summed over their appropriate range.

The linear operator of the QISM is

$$L(x|\lambda) = \mathrm{i}\frac{\lambda}{2}\Sigma + \Omega(x) \qquad \text{with} \qquad \Omega(x) = \mathrm{i}\sqrt{g}(\phi_j(x)E_{j,N+1} - \phi^{\dagger j}(x)E_{N+1,j}). \tag{2.2}$$

In this equation $E_{\alpha\beta}$ is the standard $(N+1)\times(N+1)$ matrix $(E_{\alpha\beta})_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu}$ and $\Sigma$ is a diagonal matrix $\Sigma = I_{N+1} - 2E_{N+1,N+1}$, where $I_{N+1}$ is the identity matrix.

The quantum monodromy matrix $T(x, y|\lambda)$ is defined by the equations

$$\frac{\partial}{\partial x} T(x,y|\lambda) =: L(x|\lambda)T(x,y|\lambda) : \qquad T(x,y|\lambda)|_{x=y} = I_{N+1} \tag{2.3}$$

where : : denotes the usual normal order for the field operators $\phi(x)$ and $\phi^\dagger(x)$. The infinite volume limit is a delicate issue in the QISM [6]. The infinite volume monodromy matrix $T(\lambda)$ is formally defined by

$$T(\lambda) = \lim_{x\to\infty, y\to-\infty} E(-x|\lambda)T(x,y|\lambda)E(y|\lambda)$$

where $E(x|\lambda) = \exp(\mathrm{i}\lambda x\Sigma/2)$. Using the implicit representation for $T(x, y|\lambda)$,

$$T(x,y|\lambda) = E(x-y|\lambda) + \int_y^x \mathrm{d}x_1\, E(x-x_1|\lambda) : \Omega(x_1)T(x_1,y|\lambda) :$$

the monodromy matrix $T(\lambda)$ can be formally computed through an iterative procedure and is expressed as

$$T(\lambda) = I_{N+1} + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}^n x\, \theta(x_1 > \cdots > x_n) E\left(2\sum_{i=1}^{n} (-)^i x_i|\lambda\right) : \Omega(x_1)\ldots\Omega(x_n) : . \tag{2.4}$$

The commutation relations for infinite volume are encoded in the exchange relation

$$R_\rho^+(\lambda-\mu)\; T(\lambda) \underset{\rho}{\otimes} T(\mu) = T(\mu) \underset{\rho}{\otimes} T(\lambda) R_\rho^-(\lambda-\mu) \tag{2.5}$$

with the $R$-matrices

$$\begin{aligned} R_\rho^\pm(\mu) = {} & \frac{\mathrm{i}\rho g}{\mu + \mathrm{i}\rho g}\mathrm{pv}\frac{1}{\mu} E_{jj} \otimes E_{kk} + \frac{1}{\mu + \mathrm{i}\rho g} E_{\alpha j} \otimes E_{j\alpha} + \frac{\mu - \mathrm{i}\rho g}{(\mu + \mathrm{i}0)^2} E_{j,N+1} \otimes E_{N+1,j} \\ & - \frac{\rho(\mu - \mathrm{i}g)}{\mu + \mathrm{i}\rho g}\mathrm{pv}\frac{1}{\mu} E_{N+1,N+1} \otimes E_{N+1,N+1} \\ & \pm \mathrm{i}\pi\delta(\mu)(E_{jj} \otimes E_{N+1,N+1} - E_{N+1,N+1} \otimes E_{jj}). \end{aligned} \tag{2.6}$$

The term i0 is a consequence of the principal value regularization adopted when $\mu$ goes to zero, according to the relation (in the sense of distributions)

$$\mathrm{pv}\frac{1}{\mu} = \frac{1}{\mu \pm \mathrm{i}0} \pm \mathrm{i}\pi\delta(\mu).$$

It is convenient to rename some elements of the monodromy matrix such as $D(\lambda) = T_{N+1,N+1}(\lambda)$ and $b^j(\lambda) = T_{N+1,j}(\lambda)$. Further examination of some components of (2.5) yields the following relations:

$$D(\lambda)D(\mu) = D(\mu)D(\lambda) \tag{2.7}$$

$$D(\lambda)b^j(\mu) = \frac{\lambda - \mu + \mathrm{i}g}{\lambda - \mu + \mathrm{i}0} b^j(\mu)D(\lambda) \tag{2.8}$$

$$b^j(\lambda)b^k(\mu) = \frac{-\rho(\lambda - \mu)}{\lambda - \mu - \mathrm{i}g} b^k(\mu)b^j(\lambda) - \frac{\mathrm{i}g}{\lambda - \mu - \mathrm{i}g} b^j(\mu)b^k(\lambda). \tag{2.9}$$

The matrix element $D(\lambda)$ serves as a generating operator-function for the commuting integrals of motion of the NLS model. This is most easily seen by performing an asymptotic expansion for large $\lambda$ in the solution (2.4), whereby one gets

$$\begin{aligned} D(\lambda) = 1 + \frac{\mathrm{i}g}{\lambda}\hat{N} + \frac{\mathrm{i}g}{\lambda^2}\left(P + \mathrm{i}\frac{g}{2}\hat{N}(\hat{N} - 1)\right) \\ + \frac{\mathrm{i}g}{\lambda^3}\left(H + \mathrm{i}g(\hat{N} - 1)P - \frac{g^2}{6}\hat{N}(\hat{N} - 1)(\hat{N} - 2)\right) + \mathrm{O}\left(\frac{1}{\lambda^4}\right) \end{aligned} \tag{2.10}$$

with

$$\hat{N} = \int \mathrm{d}x\, \phi^{\dagger j}\phi_j \qquad P = -\mathrm{i}\int \mathrm{d}x\, \phi^{\dagger j}\partial\phi_j$$

and $H$ is given in (2.1). Consequently, equation (2.7) implies that these integrals of motion are all in involution.

In the multicomponent NLS model the commutation relations amongst $b^j(\lambda)$ and their adjoint have to be deduced from another type of exchange relation (see [5] for details). Moreover, these operators hardly make sense as operators on the Hilbert space [6] and it is necessary to consider the scattering states operators instead

$$a^{\dagger j}(\lambda) = \frac{\mathrm{i}}{\sqrt{g}} b^j(\lambda)D^{-1}(\lambda) \tag{2.11}$$

and their adjoint $a_k(\lambda)$. Then the commutation relations amongst $a(\lambda)$ and $a^\dagger(\lambda)$ are nicely encoded in the form of a ZF algebra [4] as (a sort of deformed oscillator algebra):

$$a_j(\lambda)a_k(\mu) = R^{nm}_{kj}(\mu - \lambda)a_m(\mu)a_n(\lambda) \tag{2.12}$$

$$a^{\dagger j}(\lambda)a^{\dagger k}(\mu) = a^{\dagger m}(\mu)a^{\dagger n}(\lambda)R^{jk}_{mn}(\mu - \lambda) \tag{2.13}$$

$$a_j(\lambda)a^{\dagger k}(\mu) = a^{\dagger m}(\mu)R^{kn}_{jm}(\lambda - \mu)a_n(\lambda) + \delta^k_j\delta(\lambda - \mu) \tag{2.14}$$

where the $R$-matrix given by

$$R^{kn}_{jm}(\mu) = \frac{1}{\mu + \mathrm{i}g}(-\rho\mu\delta^k_m\delta^n_j + \mathrm{i}g\delta^k_j\delta^n_m) \tag{2.15}$$

is the two-body scattering matrix of the $N$-component NLS model.

The operators $a^\dagger(\lambda)$ and $a(\lambda)$ play the role of creation and annihilation operators and as such can be used to build a Fock space through their action on the vacuum. Owing to (2.8) these states are simultaneous eigenstates of the conserved quantities. Actually, the set of (asymptotic scattering) states $\prod_{i=1}^{n} a^{\dagger i}(\lambda_i)|0\rangle$ for all $n$ is dense in the Hilbert space of the NLS model [6, 7]. This property will be useful later on.

The original field operator $\phi_k(x)$ can be recovered from the knowledge of the scattering data. This is achieved by solving a system of quantum Gel'fand–Levitan equations, whose result is a quantum version of the Rosales expression [8, 9] (given here for $t = 0$):

$$\phi_k(x) = \sum_{n=0}^{\infty}(-g)^n \int \frac{\mathrm{d}^n\nu\, \mathrm{d}^{n+1}\mu}{(2\pi)^{2n+1}} a^{\dagger i_1}(\nu_1)\ldots a^{\dagger i_n}(\nu_n)a_{i_n}(\mu_n)\ldots a_{i_1}(\mu_1)a_k(\mu_0)$$
$$\times \frac{\mathrm{e}^{\mathrm{i}\mu_0 x}\prod_{l=1}^{n}\mathrm{e}^{\mathrm{i}(\mu_l-\nu_l)x}}{\prod_{l=1}^{n}(\nu_l-\mu_{l-1}+\mathrm{i}\epsilon)(\nu_l-\mu_l+\mathrm{i}\epsilon)}. \tag{2.16}$$

## 3. Yangian algebra in NLS

In this section we show that the NLS model contains an infinite set of conserved charges having the structure of a Yangian algebra. There are several equivalent definitions of the Yangian $Y(sl(N))$ and we present two of them in appendix A.

The Yangian symmetry of the NLS model already manifests itself in the exchange relation (2.5). Indeed, denoting by $\tilde{T}(\lambda)$ the $N \times N$ submatrix of $T(\lambda)$, $\tilde{T}(\lambda) = T_{ij}(\lambda)E_{ij}$, and examining the appropriate components of (2.5), one deduces the following relations:

$$\tilde{R}(\lambda-\mu)\tilde{T}(\lambda)\otimes\tilde{T}(\mu) = \tilde{T}(\mu)\otimes\tilde{T}(\lambda)\tilde{R}(\lambda-\mu) \tag{3.1}$$

with yet another $R$-matrix

$$\tilde{R}(\lambda-\mu) = (\lambda-\mu)E_{jk}\otimes E_{kj} + \mathrm{i}\rho g I_N \otimes I_N. \tag{3.2}$$

This coincides precisely with the defining relation of the Yangian $Y(gl(N))$.

The fact that the Yangian algebra commutes with the Hamiltonian of the NLS model is a consequence of the exchange relation as well, since one extracts from (2.5) that

$$[\tilde{T}_{ij}(\lambda), D(\mu)] = 0 \tag{3.3}$$

and the Hamiltonian is just one of the conserved quantities in the asymptotic expansion (2.10).

It is of some interest to obtain an explicit representation of the Yangian generators in terms of the field operators $\phi_i(x)$. This is achieved with the help of the iterative solution (2.4) of the monodromy matrix $T(\lambda)$, and by looking at its asymptotic expansion for large $\lambda$ one finds that

$$\tilde{T}_{jk}(\lambda) = \delta_{jk} + \frac{\mathrm{i}\rho g}{\lambda}\int \mathrm{d}x\, \phi^{\dagger k}(x)\phi_j(x) + \frac{\rho g}{\lambda^2}\left(\int \mathrm{d}x\, \phi^{\dagger k}(x)\partial\phi_j(x)\right.$$
$$\left. + g\int \mathrm{d}^2x\, \theta(x_1 > x_2)\phi^{\dagger n}(x_1)\phi^{\dagger k}(x_2)\phi_j(x_1)\phi_n(x_2)\right) + \mathrm{O}\left(\frac{1}{\lambda^3}\right) \tag{3.4}$$
$$\equiv \delta_{jk} + \mathrm{i}\rho g\sum_{n=0}^{\infty}\frac{\tilde{T}_{jk}^{(n)}}{\lambda^{n+1}}. \tag{3.5}$$

In particular, this shows that in this model the formal series expansion of the Yangian generators (see (A.2)) is to be understood as an asymptotic expansion for large $\lambda$.

The most important relations for our purpose are the commutators of the Yangian generators $\tilde{T}_{jl}^{(n)}$ with the creation operators $a^{\dagger k}(\mu)$. Using the exchange relation (2.5), definition (2.11) and the symmetry property (3.3) one finds that

$$[\tilde{T}_{jl}(\lambda), a^{\dagger k}(\mu)] = \frac{\mathrm{i}\rho g}{\lambda-\mu-\mathrm{i}0}a^{\dagger l}(\mu)\tilde{T}_{jk}(\lambda) \tag{3.6}$$

which, upon expanding in $1/\lambda$, yields

$$
\begin{aligned}
&[\tilde{T}^{(0)}_{jl}, a^{\dagger k}(\mu)] = a^{\dagger l}(\mu)\delta_{jk}, \\
&[\tilde{T}^{(1)}_{jl}, a^{\dagger k}(\mu)] = \mu a^{\dagger l}(\mu)\delta_{jk} + \mathrm{i}\rho g a^{\dagger l}(\mu)\tilde{T}^{(0)}_{jk}.
\end{aligned}
\tag{3.7}
$$

In the next section we need a more convenient basis of $Y(sl(N))$ (see appendix A). It is also convenient to deal with self-adjoint generators for the Yangian algebra. It turns out that these two requirements can be fulfilled in a single operation. Let us denote by $\tilde{T}(\lambda)^\ddagger$ the Hermitian conjugate of the $N \times N$ matrix $\tilde{T}(\lambda)$ obtained by transposing the matrix and taking the adjoint of its entries (which are operators in a Hilbert space), namely

$$
\tilde{T}(\lambda)^\ddagger = I_N - \mathrm{i}\rho g \sum_{n=0}^{\infty} \frac{\tilde{T}^{(n)\dagger}_{jk} E_{kj}}{\lambda^{n+1}}. \tag{3.8}
$$

Its anti-Hermitian part is simply

$$
\frac{1}{2}(T(\lambda) - T(\lambda)^\ddagger) = \mathrm{i}\rho g \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \frac{1}{2}(\tilde{T}^{(n)}_{jk} + \tilde{T}^{(n)\dagger}_{kj}) E_{jk} \equiv \mathrm{i}\rho g U(\lambda) \tag{3.9}
$$

where $U(\lambda) = U(\lambda)^\ddagger$ is now Hermitian. As such, it can be expanded as

$$
U(\lambda) = \sum_{n=0}^{\infty} \frac{U^{(n)}_{jk} E_{jk}}{\lambda^{n+1}} = \sum_{n=0}^{\infty} \frac{\sum_{a=1}^{N^2-1} \tilde{Q}^a_n t_a + \tilde{Q}^0_n I_N}{\lambda^{n+1}} \tag{3.10}
$$

where $\tilde{Q}^a_n = \tilde{Q}^{a\dagger}_n$ are self-adjoint generators. The matrices $t_a = (t_a)^{*T}$ are in the fundamental representation of $su(N)$ and normalized to

$$
[t_a, t_b] = \mathrm{i} f^c_{ab} t_c \qquad \eta_{ab} = \mathrm{tr}(t_a t_b).
$$

Therefore, we have that

$$
\tilde{Q}^a_n = \mathrm{tr}(U^{(n)} t^a) = U^{(n)}_{jk} (t^a)_{kj}. \tag{3.11}
$$

In the $\tilde{Q}^a_n$ basis, only the $n = 0, 1$ grades are necessary and from (3.4) we find that

$$
\begin{aligned}
U^{(0)}_{jk} &= \tilde{T}^{(0)}_{jk} \\
U^{(1)}_{jk} &= \tilde{T}^{(1)}_{jk} - \frac{\mathrm{i}\rho g}{2} \tilde{T}^{(0)}_{lk} \tilde{T}^{(0)}_{jl} + \frac{\mathrm{i}\rho g N}{2} \tilde{T}^{(0)}_{jk}.
\end{aligned}
\tag{3.12}
$$

The commutation relations of $\tilde{Q}^a_n$ and $a^{\dagger k}(\mu)$ are then readily computed using (3.7), (3.11) and (3.12)

$$
[\tilde{Q}^a_0, a^{\dagger k}(\mu)] = (a^\dagger(\mu) t^a)^k \tag{3.13}
$$

$$
[\tilde{Q}^a_1, a^{\dagger k}(\mu)] = \mu\, (a^\dagger(\mu) t^a)^k - \frac{\rho g}{2} f^a_{bc} (a^\dagger(\mu) t^c)^k \tilde{Q}^b_0. \tag{3.14}
$$

The operators $\tilde{Q}^a_{0,1}$ generates the Yangian $Y(sl(N))$ and the operators $\tilde{Q}^0_n$ are related to its centre. They are also connected in some intricate way to the integrals of motion.

It is instructive to study the type of Yangian representations that appear in the Hilbert space of the NLS model. The vacuum is invariant under the action of $\tilde{Q}^a_{0,1}$ and the one-particle state $a^{\dagger k}(\mu)|0\rangle$ transforms as an evaluation representation in the fundamental representation of $sl(N)$ [10], denoted by $V(\mu)$, since

$$
\begin{aligned}
\tilde{Q}^a_0 a^{\dagger k}(\mu)|0\rangle &= (t^a)_j{}^k a^{\dagger j}(\mu)|0\rangle \\
\tilde{Q}^a_1 a^{\dagger k}(\mu)|0\rangle &= \mu (t^a)_j{}^k a^{\dagger j}(\mu)|0\rangle.
\end{aligned}
$$

The two-particle state, $a^{\dagger k_1}(\mu_1)a^{\dagger k_2}(\mu_2)|0\rangle$, transforms as a tensor product of two such representations

$$\begin{aligned}
\tilde{Q}_0^a a^{\dagger k_1}(\mu_1)a^{\dagger k_2}(\mu_2)|0\rangle &= ((t^a)_{j_1}{}^{k_1}\delta_{j_2}^{k_2} + \delta_{j_1}^{k_1}(t^a)_{j_2}{}^{k_2})a^{\dagger j_1}(\mu_1)a^{\dagger j_2}(\mu_2)|0\rangle \\
\tilde{Q}_1^a a^{\dagger k_1}(\mu_1)a^{\dagger k_2}(\mu_2)|0\rangle &= \Big(\mu_1(t^a)_{j_1}{}^{k_1}\delta_{j_2}^{k_2} + \mu_2\delta_{j_1}^{k_1}(t^a)_{j_2}{}^{k_2} \\
&\quad - \frac{\rho g}{2} f_{bc}^a (t^c)_{j_1}{}^{k_1}(t^b)_{j_2}{}^{k_2}\Big)a^{\dagger j_1}(\mu_1)a^{\dagger j_2}(\mu_2)|0\rangle.
\end{aligned} \tag{3.15}$$

In particular, the second term on the right-hand side of (3.14) ensures that the action of $\tilde{Q}_1^a$ on the tensor product is consistent with the comultiplication of the Yangian

$$\begin{aligned}
\Delta(\tilde{Q}_0^a) &= \tilde{Q}_0^a \otimes 1 + 1 \otimes \tilde{Q}_0^a \\
\Delta(\tilde{Q}_1^a) &= \tilde{Q}_1^a \otimes 1 + 1 \otimes \tilde{Q}_1^a - \frac{\rho g}{2} f_{bc}^a \tilde{Q}_0^c \otimes \tilde{Q}_0^b.
\end{aligned}$$

Therefore, $n$-particle states will carry an $n$-fold tensor product of $V(\mu_i)$ representations.

## 4. Yangian generators and deformed oscillators

In view of formulae (2.16) and (3.4), it is clear that trying to reconstruct the Yangian generators (in term of oscillators) by direct calculation is a difficult task. Instead we define two operators $Q_0^a$ and $Q_1^a$ that have the same commutation relations with $a^\dagger(\mu)$ as in (3.13) and (3.14). Therefore, their action on the Fock space spanned by the $a^\dagger(\mu)$ will coincide with that of $\tilde{Q}_{0,1}^a$ and, as this Fock space is dense in the Hilbert space of the NLS model, we shall identify these operators. All the other Yangian generators are built from these two sets of elements.

In order to simplify the presentation, we adopt a more compact notation for the ZF algebra. We drop the explicit mention of the indices $i$, $j$ and the momenta $\mu_i$, and instead introduce a new index referring to an $N$-dimensional auxiliary space. More explicitly

$$a_1 \equiv a_i(\mu_1)e_1^i$$

where $e_1^i$ is some basis of the $N$-dimensional auxiliary space labelled by 1. For instance, the $R$-matrix (2.15) reads

$$R_{12} \equiv R_{12}(\mu_1 - \mu_2) = \frac{1}{\mu_1 - \mu_2 + \mathrm{i}g}(-\rho(\mu_1 - \mu_2)1 \otimes 1 + \mathrm{i}gP_{12}) \tag{4.1}$$

where $P_{12}$ is the permutation operator in the auxiliary spaces. The inverse of $R_{12}$ is $R_{21} = R_{21}(\mu_2 - \mu_1)$. With this notation, the ZF algebra relations read

$$\begin{aligned}
a_1 a_2 &= R_{21} a_2 a_1 \\
a_1^\dagger a_2^\dagger &= a_2^\dagger a_1^\dagger R_{21} \\
a_1 a_2^\dagger &= a_2^\dagger R_{12} a_1 + \delta_{12}.
\end{aligned} \tag{4.2}$$

We also rename the operators we are looking for as $J^a = Q_0^a$ and $S^a = Q_1^a$. Thus (3.13) and (3.14), and their conjugate, translate to

$$[J^a, a_0^\dagger] = a_0^\dagger t_0^a \tag{4.3}$$

$$[J^a, a_0] = -t_0^a a_0 \tag{4.4}$$

$$[S^a, a_0^\dagger] = \mu_0 a_0^\dagger t_0^a + \frac{\rho g}{2} f^a{}_{bc} a_0^\dagger t_0^b J^c \tag{4.5}$$

$$[S^a, a_0] = -\mu_0 t_0^a a_0 + \frac{\rho g}{2} f^a{}_{bc} J^b t_0^c a_0. \tag{4.6}$$

Here $t_0^a$ means that the $su(N)$ matrix $t^a$ is acting in the auxiliary space labelled by 0.

We first construct the operator $J^a$. The idea is to build it recursively in such a way that relations (4.3), (4.4) are fulfilled. The expansion parameter is not the coupling constant $g$ but rather the number of oscillators. More precisely, we start with the expression

$$J^a = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} J^a_{(n)} = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} a^\dagger_{1\ldots n} T^a_{1\ldots n} a_{n\ldots 1} \tag{4.7}$$

where $a_{n\ldots 1} = a_n(\mu_n)\ldots a_2(\mu_2)a_1(\mu_1)$ and the integration on $\mu_1, \mu_2, \ldots, \mu_n$ is implied in $J^a$. We then determine the tensors $T^a_{1\ldots n}$ recursively. The details of the calculation are relegated in appendix B, and as a result we find that

$$T^a_{1\ldots n} = \sum_{j=1}^{n} \alpha^n_j\, t^a_j \qquad \text{with} \quad \alpha^n_j = (-)^{j-1}\begin{pmatrix} n-1 \\ j-1 \end{pmatrix}. \tag{4.8}$$

These generators also verify

$$[J^a, J^b] = \mathrm{i} f^{ab}_c J^c. \tag{4.9}$$

They form the $sl(N)$ subalgebra of $Y(sl(N))$.

We then look for operators $S^a$ of a similar form

$$S^a = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} S^a_{(n)} = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} a^\dagger_{1\ldots n} \tilde{T}^a_{1\ldots n} a_{n\ldots 1} \tag{4.10}$$

satisfying (4.5), (4.6). In this case, the procedure is simpler since $S^a$ lives in the adjoint representation of the subalgebra generated by $J^a$. This implies that

$$S^a = -\frac{\mathrm{i}}{c_2} f^a_{bc}[S^b, J^c] \tag{4.11}$$

where $c_2$ is, as usual, the second Casimir in the adjoint representation, $c_2\delta^b_a = f_{acd} f^{bcd}$. Using the explicit expression (4.7) for $J^a$ and imposing that $S^a$ satisfies (4.5), (4.6) enables us to compute the right-hand side of (4.11) and to determine the tensors $\tilde{T}^a_{1\ldots n}$.

As anticipated from (4.5) the tensors depend on the momenta and their expression is (see appendix C):

$$\tilde{T}^a_{1\ldots n} = \sum_{j=1}^{n} \alpha^n_j \left( \mu_j t^a_j + \frac{\rho g}{2} f^a{}_{bc} \sum_{i=1}^{j-1} t^b_i t^c_j \right). \tag{4.12}$$

Therefore, expressions (4.7) and (4.10) provide a realization of the Yangian generators in terms of the ZF algebra.

A similar procedure can be applied to the integrals of motion (or higher Hamiltonians) of the NLS model. The lowest Hamiltonians $\hat{N}, P, H$ are explicitly known, and from (2.8), (2.10) and (2.11) we find that

$$[\hat{N}, a_0^\dagger] = a_0^\dagger$$
$$[P, a_0^\dagger] = \mu_0 a_0^\dagger$$
$$[H, a_0^\dagger] = \mu_0^2 a_0^\dagger.$$

Let us now define the infinite set of commuting operators $\hat{I}_n$, $n \geqslant 0$ by

$$\hat{I}_n = \int \mathrm{d}\mu_1\, \mu_1^n a_1^\dagger a_1 \tag{4.13}$$

which enjoy the following commutation relations:

$$\begin{aligned} [\hat{I}_n, a_0^\dagger] &= \mu_0^n a_0^\dagger \\ [\hat{I}_n, a_0] &= -\mu_0^n a_0. \end{aligned} \tag{4.14}$$

This implies that acting on an $m$-particle state

$$\hat{I}_n a_1^\dagger \dots a_m^\dagger |0\rangle = \left( \sum_{j=1}^{m} \mu_j^n \right) a_1^\dagger \dots a_m^\dagger |0\rangle \tag{4.15}$$

which is precisely the definition of the higher Hamiltonians in the quantum NLS model [6]. According to (4.14) the lowest ones are obviously identified with

$$\hat{N} = \hat{I}_0 \qquad P = \hat{I}_1 \qquad H = \hat{I}_2.$$

Moreover, we can show that $D(\lambda)$ is a generating operator-function for the integrals of motion, that is, it can be expressed entirely in terms of the $\hat{I}_n$ operators. Indeed we can prove that

$$D(\lambda) = \exp(d(\lambda)) \qquad \text{where} \quad d(\lambda) = \sum_{n=0}^{\infty} \frac{d_n}{\lambda^{n+1}} \tag{4.16}$$

and

$$d_n = \mathrm{i}g \sum_{j=0}^{n} \frac{(-\mathrm{i}g)^{n-j}}{n+1} \binom{n+1}{j} \hat{I}_j. \tag{4.17}$$

As the $\hat{I}_n$ commute with each other and satisfy (4.14), it is straightforward to show that

$$\exp(d(\lambda)) a_0^\dagger \exp(-d(\lambda)) = \left( 1 + \frac{\mathrm{i}g}{\lambda - \mu_0)} \right) a_0^\dagger$$

which is precisely the relation between $D(\lambda)$ and $a_0^\dagger$ as deduced from (2.8). This proves the assertion (4.16).

Owing to the explicit expressions (4.7) and (4.10), it is very easy to check that the operators $\hat{I}_n$ commute with $J^a$ and $S^a$. As $J^a$, $S^a$ generate the Yangian algebra, then obviously the $\hat{I}_n$ commute with the whole Yangian algebra. This is just another way of expressing the content of (3.3). It also means that the Yangian is a symmetry of all the quantum systems defined with the help of the higher Hamiltonians.

As usual, the time evolution of the quantum field $\phi_k(x)$ of equation (2.16) is given by the conjugation due to the NLS Hamiltonian (2.1) which is nothing but $\hat{I}_2$. According to the commutation relations (4.14), this amounts to multiplying in (2.16) the creation operators $a^\dagger(\nu)$ by $\mathrm{e}^{\mathrm{i}\nu^2 t}$ and the annihilation operators $a(\mu)$ by $\mathrm{e}^{-\mathrm{i}\mu^2 t}$. Although expression (2.16) was originally obtained as the solution to the NLS equation of motion, it also provides the solution to the higher flows of the hierarchy. Simply, the $t_n$-time evolution is now induced by a conjugation by $\exp(\mathrm{i}\hat{I}_n t_n)$. Consequently, the phases multiplying the creation and annihilation operators are, respectively, $\mathrm{e}^{\mathrm{i}\nu^n t_n}$ and $\mathrm{e}^{-\mathrm{i}\mu^n t_n}$. It is remarkable to obtain the solutions to all the equations of motion of the hierarchy in such an easy way.

## 5. Connection with finite $W$-algebras

It has already been shown that there is a strong connection between Yangians $Y(sl(N))$ and finite $W$ algebras [11]. In this section we show that the NLS hierarchy offers a natural framework to illustrate this relation.

For such a purpose, we focus on the Fock space $\mathcal{F}$ spanned by the $a^{i\dagger}(\mu)$ and detail its structure. Let us recall that the $p$-particle subspace $\mathcal{F}_p(\mu_1, \dots, \mu_p)$ with fixed momenta $(\mu_1, \dots, \mu_p)$ is a tensor product of $p$ evaluation representations $\otimes_{i=1}^{p} V(\mu_i)$, all in the fundamental representation of $su(N)$. The $p$-particles subspace $\mathcal{F}_p$ is just the span over all momenta $\mu_i$.

It is straightforward to see that on $\mathcal{F}_p$ the Yangian generators $t_{ij}^{(n)}$ with $n \geqslant p$ act as zero operators. We thus have a representation of a truncated Yangian, which is known to be isomorphic, at the algebra level, to a finite $W(gl(Np), N \cdot sl(p))$ algebra [12]. Thus on each $p$-particle subspace the Yangian acts as a $W(gl(Np), N \cdot sl(p))$ algebra. This is another nice application of finite $W$ algebras (more examples can be found in [13] and references therein).

Let us illustrate this point in the simplest case, namely $p = 2$ and $N = 2$. Consequently, the only independent Yangian generators are $Q_0^a$, $Q_1^a$ and their action on $\mathcal{F}_2$ is given in (3.15). Let us denote their representation on $\mathcal{F}_2$ by $J^a$ and $S^a$, respectively. Up to an innocuous shift $S^a \to S^a - \frac{1}{2} P J^a$, the full set of commutation relations satisfied by these operators is

$$\begin{aligned} [J^a, J^b] &= \mathrm{i} f_c^{ab} J^c \\ [J^a, S^b] &= \mathrm{i} f_c^{ab} S^c \\ [S^a, S^b] &= \mathrm{i} f_c^{ab} \left( \frac{1}{2} H - \frac{1}{4} P^2 - g^2 \frac{c_2}{8N} \right) J^c. \end{aligned} \tag{5.1}$$

One recognizes here the relations of the $W$-algebra $W(gl(4), 2 \cdot sl(2))$.

$W(gl(4), 2 \cdot sl(2))$ and $W(sl(4), 2 \cdot sl(2))$ algebras essentially differ by one central element. In the present context, this element is nothing but $P$, the total momentum. Thus, the transition between the two $W$-algebras amounts to describing the system in its centre of mass frame.

## 6. Conclusions and outlook

We deliberately imposed the restriction that the coupling constant $g$ be positive. When $g$ is negative, the quantum spectrum also contains bound states (solitons) and the asymptotic scattering states $\prod_{i=1}^{n} a^{\dagger i}(\lambda_i)|0\rangle$ are no longer complete in the Hilbert space of the NLS model. Our construction relies crucially on the completeness of those states in order to identify the generators of the Yangian symmetry (3.11) with the generators (4.7) and (4.10) expressed in term of the ZF algebra elements.

In both regions of the coupling constant, the Yangian generators commute with the scattering matrix and the Yangian corresponds to a symmetry of the scattering matrix. The nuance is that in the second situation we have only defined the action of the Yangian generators on the asymptotic scattering states and the best we can say about the operators (4.7) and (4.10) is that they generate an asymptotic symmetry of the NLS model. Extending their definition to the full Hilbert space would require complete knowledge of the bound states spectrum of the NLS model, something not yet achieved.

From a more general point of view, when considering our construction as based solely on the existence of a ZF algebra we may conclude that it is possible to realize a Yangian algebra on the Fock space generated by the ZF algebra whenever the $R$-matrix is of the rational type given in (2.15). For the NLS model, the $S$-matrix is invariant under $SU(N)$ and the symmetry algebra turns out to be a Yangian $Y(sl(N))$. Let us mention that the $SU(N)$-Thirring model is another quantum system with such rational $R$-matrix, thus providing a relativistic example of a system with Yangian symmetry [14].

It would be interesting to exhibit the symmetry algebra of more general cases where the $R$-matrix is not rational but is still invariant under some Lie group. As the $R$-matrix in the ZF algebra can be interpreted as the two-body $S$-matrix of integrable systems in 1 + 1 dimensions, this question reduces to the problem of identifying the largest $S$-matrix symmetry of such integrable systems. The study of this sort of relationship in a more general situation is certainly interesting and we intend to return to it in a future paper.

The NLS model considered in the present work is defined on the line. Usually the presence of boundaries strongly influences the symmetry and we are currently investigating this issue in the NLS model on the half line. Its quantification as a quantum field theory, as carried out in [15], reveals the presence of a boundary exchange algebra generalizing the ZF algebra, thus fitting well within the more general scheme of integrable systems with boundaries developed in [16]. We still expect the system to possess a large internal symmetry, possibly in the form of a twisted Yangian algebra [17] and we also expect to be able to express these symmetry generators in terms of the boundary exchange algebra [18].

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## Appendix A. Definition of $Y(gl(N))$

The Yangian $Y(gl(N))$ can be defined as the free associative algebra over $\mathbb{C}$ with generators $1, t_{ij}^{(n)}, n \geqslant 0$ (not to be confused with $t_k^a$ used elsewhere in the text) quotiented by the relation [10]

$$R_Y(\lambda - \mu)\, t(\lambda) \otimes t(\mu) = t(\mu) \otimes t(\lambda)\, R_Y(\lambda - \mu) \tag{A.1}$$

where we introduced the $N \times N$ matrix $t(\lambda)$ whose entries are formal series in $\lambda$

$$t_{ij}(\lambda) = \delta_{ij} + h \sum_{n=0}^{\infty} \frac{t_{ij}^{(n)}}{\lambda^{n+1}} \tag{A.2}$$

and the $R$-matrix is given by

$$R_Y(\lambda - \mu) = (\lambda - \mu) E_{ij} \otimes E_{ji} - h I_N \otimes I_N. \tag{A.3}$$

The non-zero deformation parameter $h$ can be scaled away as two Yangians with different non-zero deformation parameter are known to be isomorphic.

The quantum determinant

$$\det_q(t(\lambda)) = \sum_{\pi \in S_N} \mathrm{sign}(\pi) t_{1,\pi(1)}(\lambda - N + 1) \ldots t_{N,\pi(N)}(\lambda) \tag{A.4}$$

generates the infinite-dimensional centre $\mathcal{Z}$, and the Yangian $Y(gl(N))$ is isomorphic to $\mathcal{Z} \otimes Y(sl(N))$.

The coproduct in this presentation is simply

$$\Delta(t_{ij}(\lambda)) = \sum_{k=1}^{N} t_{ik}(\lambda) \otimes t_{kj}(\lambda). \tag{A.5}$$

In the main text, we use the notion of an evaluation representation. In the $t_{ij}^{(n)}$ basis, it is defined by the composition of the algebra homomorphism

$$t_{ij}(\lambda) = \delta_{ij} + \frac{e_{ij}}{\lambda} \tag{A.6}$$

where $e_{ij}$ are the generators of $gl(N)$, with any representation of $gl(N)$. In particular, the Yangian generators $t_{ij}^{(n)}$ with $n \geqslant 1$ act as zero operators.

Alternatively, the Yangian $Y(sl(N))$ can also be defined as the unique homogeneous quantization of $sl(N)[u]$ (the polynomial maps from the complex plane to $sl(N)$) [19]. It is generated by the two sets of elements $Q_0^a$ (a basis of $sl(N)$) and $Q_1^a$ subject to the following constraints:

$$[Q_0^a, Q_n^b] = \mathrm{i} f_c^{ab} Q_n^c$$
$$[Q_1^a, [Q_0^b, Q_1^c]] + [Q_1^b, [Q_0^c, Q_1^a]] + [Q_1^c, [Q_0^a, Q_1^b]] = h^2 f^a{}_{pd} f^b{}_{qx} f^c{}_{ry} f^{xy}{}_e \kappa^{de} s_3(Q_0^p, Q_0^q, Q_0^r)$$
$$[[Q_1^a, Q_1^b], [Q_0^c, Q_1^d]] + [[Q_1^c, Q_1^d], [Q_0^a, Q_1^b]] = h^2 (f^a{}_{pe} f^b{}_{qx} f^{cd}{}_y f^y{}_{rz} f^{xz}{}_g + f^c{}_{pe} f^d{}_{qx} f^{ab}{}_y f^y{}_{rz} f^{xz}{}_g) \kappa^{eg} s_3(Q_0^p, Q_0^q, Q_1^r)$$

where $\kappa^{ab}$ is the Killing form on $sl(N)$ and $s_3(.,.,.)$ is the totally symmetrized product of three terms (normalized to $s_3(x,x,x) = x^3$).

The first presentation is helpful to identify the type of algebraic structure generated by $\tilde{T}(\lambda)$ in (3.1). The second one is more convenient when explicitly constructing the Yangian generators, as only two sets of elements are necessary to generate the whole Yangian $Y(sl(N))$.

## Appendix B. Construction of $J^a$

To construct $J^a$, we heavily use the notation of internal spaces, which encodes both $su(N)$ indices and momenta. Recall also, that in this notation, $R_{ji} = R_{ij}^{-1}$. The key observation in the construction of $J^a$ is that, due to the presence of $\delta_{ij}$ in (4.2), the commutator of $J_{(n)}^a$ with $a_0^\dagger$ contains two contributions with different numbers of oscillators. For the simplest case of $n = 1$ one has

$$[J_{(1)}^a, a_0^\dagger] = a_0^\dagger a_1^\dagger (R_{01} T_1^a R_{10} - T_1^a) a_1 + a_0^\dagger T_0^a. \tag{B.1}$$

Comparing with (4.3), the term with no annihilation operator completely fixes $T_1^a = t_1^a$. In the commutator $[J^a, a_0^\dagger]$, the only other term with one annihilation operator comes from the commutator $[J_{(2)}^a, a_0^\dagger]$ and we define $T_{12}^a$ so that these two contributions cancel. Repeating this procedure for increasing number of oscillators uniquely determines the tensors $T_{1\ldots n}^a$.

In this appendix, we adopt a different point of view, namely we prove that the solution given in (4.8) is indeed correct.

For generic $n$, one has

$$[J_{(n)}^a, a_0^\dagger] = a_0^\dagger a_{1\ldots n}^\dagger \left(\mathcal{R}_n^{-1} T_{1\ldots n}^a \mathcal{R}_n - T_{1\ldots n}^a\right) a_{n\ldots 1} + a_0^\dagger a_{1\ldots n-1}^\dagger B_n a_{n-1\ldots 1} \tag{B.2}$$

where

$$B_n = \sum_{i=1}^{n} \mathcal{R}_{i-1}^{-1} T_{1\ldots n|i}^a \mathcal{R}_{i-1} \tag{B.3}$$
$$\mathcal{R}_i = R_{i0} \ldots R_{20} R_{10} \tag{B.4}$$
$$\mathcal{R}_i^{-1} = R_{01} R_{02} \ldots R_{0i}. \tag{B.5}$$

Here the notation $T_{1\ldots n|i}^a$ represents the indices substitutions $i \to 0$ and $k \to k-1$ for $k > i$.

Then, we simplify the expression for $B_n$:

$$\begin{aligned} B_n &= \sum_{i=1}^{n} \left( \mathcal{R}_{i-1}^{-1} \left( \sum_{k=1}^{i-1} \alpha_k^n t_k^a + \alpha_i^n t_0^a + \sum_{k=i}^{n-1} \alpha_{k+1}^n t_k^a \right) \mathcal{R}_{i-1} \right) \\ &= \sum_{i=1}^{n} \left( \sum_{k=1}^{i-1} \alpha_k^n \mathcal{R}_{n-1}^{-1} t_k^a \mathcal{R}_{n-1} + \alpha_i^n \mathcal{R}_{i-1}^{-1} t_0^a \mathcal{R}_{i-1} + \sum_{k=i}^{n-1} \alpha_{k+1}^n t_k^a \right) \\ &= \mathcal{R}_{n-1}^{-1} \left( \sum_{0 \leqslant k < i \leqslant n} \alpha_k^n t_k^a \right) \mathcal{R}_{n-1} + \sum_{1 \leqslant i \leqslant k \leqslant n-1} \alpha_{k+1}^n t_k^a + C_n \end{aligned}$$

where $C_n$ is defined as

$$C_n = \sum_{i=1}^{n} \alpha_i^n \mathcal{R}_{i-1}^{-1} t_0^a \mathcal{R}_{i-1}.$$

Next, performing the independent sums on $i$ and using the properties

$$\alpha_k^n = \frac{n-1}{n-k}\alpha_k^{n-1} \qquad \alpha_{k+1}^n = -\frac{n-1}{k}\alpha_k^{n-1}$$

one gets

$$\begin{aligned} B_n - C_n &= \mathcal{R}_{n-1}^{-1} \sum_{k=1}^{n-1} (n-k)\alpha_k^n t_k^a \mathcal{R}_{n-1} + \sum_{k=1}^{n-1} k\, \alpha_{k+1}^n t_k^a \\ &= (n-1)\mathcal{R}_{n-1}^{-1} \left( \sum_{k=1}^{n-1} \alpha_k^{n-1} t_k^a \right) \mathcal{R}_{n-1} - (n-1) \sum_{k=1}^{n-1} \alpha_{k+1}^{n-1} t_k^a \\ &= (n-1)\mathcal{R}_{n-1}^{-1} T_{1\ldots n-1}^a \mathcal{R}_{n-1} - (n-1) T_{1\ldots n-1}^a. \end{aligned}$$

The simplification of $C_n$ is achieved using the $sl(N)$ invariance of the $R$-matrix, $[R_{0k}, t_0^a + t_k^a] = 0$, and the properties

$$\sum_{i=1}^{n} \alpha_i^n = 0 \qquad \sum_{i=k+1}^{n} \alpha_i^n = -\alpha_k^{n-1}$$

and leads to

$$\begin{aligned} C_n &= \sum_{i=1}^{n} \alpha_i^n [\mathcal{R}_{i-1}^{-1}, t_0^a] \mathcal{R}_{i-1} + \sum_{i=1}^{n} \alpha_i^n t_0^a = \sum_{i=1}^{n} \alpha_i^n \sum_{k=1}^{i-1} \mathcal{R}_{k-1}^{-1} [R_{0k}, t_0^a] \mathcal{R}_k \\ &= \sum_{k=1}^{n-1} \left( \sum_{i=k+1}^{n} \alpha_i^n \right) \mathcal{R}_{k-1}^{-1} [R_{0k}, t_0^a] \mathcal{R}_k = \sum_{k=1}^{n-1} \alpha_k^{n-1} \mathcal{R}_{k-1}^{-1} [R_{0k}, t_k^a] \mathcal{R}_k \\ &= \mathcal{R}_{n-1}^{-1} [T_{1\ldots n-1}^a, \mathcal{R}_{n-1}]. \end{aligned}$$

Finally, we get that

$$B_n = (B_n - C_n) + C_n = n\mathcal{R}_{n-1}^{-1} [T_{1\ldots n-1}^a, \mathcal{R}_{n-1}] \tag{B.6}$$

so that the commutator (B.2) reduces to

$$[J_{(n)}^a, a_0^\dagger] = a_{01\ldots n}^\dagger \mathcal{R}_n^{-1} [T_{1\ldots n}^a, \mathcal{R}_n] a_{n\ldots 1} - n a_{01\ldots n-1}^\dagger \mathcal{R}_{n-1}^{-1} [T_{1\ldots n-1}^a, \mathcal{R}_{n-1}] a_{n-1\ldots 1}.$$

Therefore, adjusting properly the coefficient of $J_{(n)}^a$ as in (4.7), we obtain a complete cancellation of all the terms but one, yielding precisely the required commutation relation (4.3). The proof of (4.4) is similar.

## Appendix C. Construction of $S^a$

We show that $S^a$ is also of the form

$$S^a = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} a_{1\ldots n}^\dagger \tilde{T}_{1\ldots n}^a a_{n\ldots 1}$$

and we determine the tensor $\tilde{T}_{1\ldots n}^a$ directly from the commutation relations

$$S^a = -\frac{\mathrm{i}}{c_2} f^a{}_{bc} [S^b, J^c] = -\frac{\mathrm{i}}{c_2} f^a{}_{bc} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} [S^b, J_{(n)}^c] \tag{C.1}$$

where $f^a{}_{bc} f^{bc}{}_d = c_2 \delta^a_d$. We require $S^a$ to satisfy (4.5) and (4.6) and consequently the right-hand side of (C.1) evaluates to

$$
\begin{aligned}
[S^b, J^c_{(n)}] &= [S^b, a^\dagger_{1\ldots n}] T^c_{1\ldots n} a_{n\ldots 1} + a^\dagger_{1\ldots n} T^c_{1\ldots n} [S^b, a_{n\ldots 1}] \\
&= \sum_{i=1}^{n} (a^\dagger_{1\ldots i-1} [S^b, a^\dagger_i] a^\dagger_{i+1\ldots n} T^c_{1\ldots n} a_{n\ldots 1} + a^\dagger_{1\ldots n} T^c_{1\ldots n} a_{n\ldots i+1} [S^b, a_i] a_{i-1\ldots 1}) \\
&= \sum_{i=1}^{n} (a^\dagger_{1\ldots i-1} \mu_i a^\dagger_i t^b_i a^\dagger_{i+1\ldots n} T^c_{1\ldots n} a_{n\ldots 1} - a^\dagger_{1\ldots n} T^c_{1\ldots n} a_{n\ldots i+1} \mu_i t^b_i a_i a_{i-1\ldots 1}) \\
&\quad + \frac{\rho g}{2} \sum_{i=1}^{n} f^b{}_{de} (a^\dagger_{1\ldots i-1} a^\dagger_i t^d_i J^e a^\dagger_{i+1\ldots n} T^c_{1\ldots n} a_{n\ldots 1} + a^\dagger_{1\ldots n} T^c_{1\ldots n} a_{n\ldots i+1} J^d t^e_i a_i a_{i-1\ldots 1})
\end{aligned}
$$

which shows that the solution we are looking for is

$$S^a = S^a_I + S^a_{II}$$

where

$$
\begin{aligned}
S^a_I &= -\frac{\mathrm{i}}{c_2} f^a{}_{bc} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} a^\dagger_{1\ldots n} \left[ \sum_{i=1}^{n} \mu_i t^b_i, T^c_{1\ldots n} \right] a_{n\ldots 1} \\
S^a_{II} &= -\frac{\mathrm{i}\rho g}{2c_2} f^a{}_{bc} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} \\
&\quad \times \sum_{i=1}^{n} f^b{}_{de} (a^\dagger_{1\ldots i} t^d_i J^e a^\dagger_{i+1\ldots n} T^c_{1\ldots n} a_{n\ldots 1} + a^\dagger_{1\ldots n} T^c_{1\ldots n} a_{n\ldots i+1} J^d t^e_i a_{i\ldots 1}).
\end{aligned}
\tag{C.2}
$$

These expressions can be simplified considerably. In $S^a_I$ we introduce the known form of $T^c_{1\ldots n} = \sum_{k=1}^{n} \alpha^n_k t^c_k$ and with the definition of the second Casimir we get

$$\mathrm{i} f^a{}_{bc} \left[ \sum_{i=1}^{n} \mu_i t^b_i, T^c_{1\ldots n} \right] = -f^a{}_{bc} \sum_{i=1}^{n} f^{bc}{}_e \mu_i \alpha^n_i t^e_i = -c_2 \sum_{i=1}^{n} \mu_i \alpha^n_i t^a_i$$

so that

$$S^a_I = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} a^\dagger_{1\ldots n} \left( \sum_{i=1}^{n} \mu_i \alpha^n_i t^a_i \right) a_{n\ldots 1}. \tag{C.3}$$

The next step is to simplify the contribution $S^a_{II}$. We shift $J^e$, $J^d$ towards $T^c$ using

$$
\begin{aligned}
[J^e, a^\dagger_{i+1\ldots n}] &= a^\dagger_{i+1\ldots n} \left( \sum_{j=i+1}^{n} t^e_j \right) \\
[J^d, a_{n\ldots i+1}] &= -\left( \sum_{j=i+1}^{n} t^d_j \right) a_{n\ldots i+1}
\end{aligned}
$$

and with the anti-symmetry of $f^b_{de}$ we get

$$
\begin{aligned}
S^a_{II} &= -\frac{\mathrm{i}\rho g}{2c_2} f^a{}_{bc} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} \\
&\quad \times \sum_{i=1}^{n} f^b{}_{de} \left( a^\dagger_{1\ldots n} \left[ \sum_{j=i+1}^{n} t^d_i t^e_j, T^c_{1\ldots n} \right] a_{n\ldots 1} + a^\dagger_{1\ldots n} [t^d_i, T^c_{1\ldots n}] J^e a_{n\ldots 1} \right).
\end{aligned}
$$

The second term is simplified using the commutator

$$\sum_{i=1}^{n} [t^d_i, T^c_{1\ldots n}] = \mathrm{i} f^{dc}_g T^g_{1\ldots n}$$

and the identity

$$f^a_{bc} f^b_{de} f^{dc}_g = -\frac{c_2}{2} f^a_{eg}.$$

That same identity, combined with the explicit expression for $T^c_{1\ldots n}$, helps to reduce the first term to

$$f^a_{bc} f^b_{de} \left[ \sum_{1 \leqslant i < j \leqslant n} t^d_i t^e_j, \sum_{k=1}^n \alpha^n_k t^c_k \right] = \mathrm{i} \frac{c_2}{2} f^a_{bc} \sum_{1 \leqslant i < j \leqslant n} (\alpha^n_i + \alpha^n_j) t^b_i t^c_j.$$

Altogether, the expression we find for $S^a_{II}$ is

$$S^a_{II} = -\frac{\rho g}{4} f^a_{bc} \sum_{n=1}^\infty \frac{(-)^n}{n!} a^\dagger_{1\ldots n} \left( \sum_{1 \leqslant i < j \leqslant n} (\alpha^n_i + \alpha^n_j) t^b_i t^c_j + T^b_{1\ldots n} J^c \right) a_{n\ldots 1}. \tag{C.4}$$

We can merge the two contributions to (C.4). We plug in the expansion (4.7) for $J^c$ in the second one and appropriately label the auxiliary spaces

$$\begin{aligned}
&\sum_{k=1}^\infty \frac{(-)^k}{k!} f^a_{bc} a^\dagger_{1\ldots k} T^b_{1\ldots k} J^c a_{k\ldots 1} \\
&\qquad = f^a_{bc} \sum_{k=1}^\infty \sum_{l=1}^\infty \frac{(-)^{k+l+1}}{k!l!} a^\dagger_{1\ldots k} a^\dagger_{k+1\ldots k+l} T^b_{1\ldots k} T^c_{k+1\ldots k+l}\, a_{k+l\ldots k+1} a_{k\ldots 1} \\
&\qquad = f^a_{bc} \sum_{n=2}^\infty \frac{(-)^{n+1}}{n!} a^\dagger_{1\ldots n} \left( \sum_{k=1}^{n-1} \binom{n}{k} T^b_{1\ldots k} T^c_{k+1\ldots n} \right) a_{n\ldots 1}
\end{aligned}$$

so that we get for $S^a_{II}$

$$S^a_{II} = -\frac{\rho g}{2} f^a_{bc} \sum_{n=2}^\infty \frac{(-)^{n+1}}{n!} a^\dagger_{1\ldots n} T^{bc}_{1\ldots n} a_{n\ldots 1} \tag{C.5}$$

where the new tensor $T^{bc}_{1\ldots n}$ is defined below and turns out to be surprisingly simple

$$\begin{aligned}
T^{bc}_{1\ldots n} &= \tfrac{1}{2} \left( \sum_{k=1}^{n-1} \binom{n}{k} T^b_{1\ldots k}\, T^c_{k+1\ldots n} - \sum_{1 \leqslant i < j \leqslant n} (\alpha^n_i + \alpha^n_j)\, t^b_i t^c_j \right) \\
&= \tfrac{1}{2} \sum_{1 \leqslant i < j \leqslant n} \left( \sum_{k=i}^{j-1} \binom{n}{k} \alpha^k_i \alpha^{n-k}_{j-k} - \alpha^n_i - \alpha^n_j \right) t^b_i t^c_j \\
&= - \sum_{1 \leqslant i < j \leqslant n} \alpha^n_j t^b_i t^c_j.
\end{aligned}$$

In the last step, we have used the property (proved in the next section)

$$\sum_{k=i}^{j-1} \binom{n}{k} \alpha^k_i \alpha^{n-k}_{j-k} - \alpha^n_i = -\alpha^n_j. \tag{C.6}$$

Putting together $S^a_I$ and $S^a_{II}$, we get the final expression for $S^a$

$$S^a = \sum_{n=1}^\infty \frac{(-)^{n+1}}{n!} a^\dagger_{1\ldots n} \left( \sum_{i=1}^n \mu_i \alpha^n_i t^a_i + \frac{\rho g}{2} f^a_{bc} \sum_{1 \leqslant i < j \leqslant n} \alpha^n_j t^b_i t^c_j \right) a_{n\ldots 1}. \tag{C.7}$$

The first few terms of this series are

$$
\begin{aligned}
S_1^a &= a_1^\dagger \mu_1 t_1^a a_1 \\
S_2^a &= -\frac{1}{2} a_{12}^\dagger \left( \mu_1 t_1^a - \mu_2 t_2^a - \frac{\rho g}{2} f^a{}_{bc} t_1^b t_2^c \right) a_{21} \\
S_3^a &= \frac{1}{6} a_{123}^\dagger \left( \mu_1 t_1^a - 2\mu_2 t_2^a + \mu_3 t_3^a - \frac{\rho g}{2} f_{bc}^a (2 t_1^b t_2^c - t_1^b t_3^c - t_2^b t_3^c) \right) a_{321}.
\end{aligned}
$$

Recall that there is an implied integration on $\mu_i$ in these expressions.

*Appendix C.1. Proof of the property (C.6)*

We want to show that for $1 \leqslant i < j \leqslant n$ we have

$$
\sum_{k=i}^{j-1} \binom{n}{k} \alpha_i^k \alpha_{j-k}^{n-k} - \alpha_i^n = -\alpha_j^n.
$$

It is equivalent to show that $f(m) = g(m)$ for $m$ integer, where

$$
\begin{aligned}
f(m) &= \sum_{k=0}^{m-1} \binom{n}{k+i} \alpha_i^{k+i} \alpha_{m-k}^{n-k-i} - \alpha_i^n \\
g(m) &= -\alpha_{i+m}^n.
\end{aligned}
$$

This is done by recursion. Obviously, $f(1) = g(1)$ and we then show that $\mathrm{d}f(m) \equiv f(m+1) - f(m)$ and $\mathrm{d}g(m) \equiv g(m+1) - g(m)$ are equal:

$$
\mathrm{d}g(m) = (-)^{i+m+1} \binom{n}{i+m}
$$

while

$$
\begin{aligned}
\mathrm{d}f(m) &= \sum_{k=0}^{m} \binom{n}{k+i} \alpha_i^{k+i} \alpha_{m-k+1}^{n-k-i} - \sum_{k=0}^{m-1} \binom{n}{k+i} \alpha_i^{k+i} \alpha_{m-k}^{n-k-i} \\
&= \sum_{k=0}^{m-1} (-)^{i+m+k+1} \binom{n}{k+i} \binom{k+i-1}{i-1} \\
&\quad \times \left\{ \binom{n-k-i-1}{m-k} + \binom{n-k-i-1}{m-k-1} \right\} \\
&\quad + (-)^{i+1} \binom{n}{m+i} \binom{m+i-1}{i-1} \\
&= \sum_{k=0}^{m-1} (-)^{i+m+k+1} \binom{n}{k+i} \binom{k+i-1}{i-1} \binom{n-k-i}{m-k} \\
&\quad + (-)^{i+1} \binom{n}{m+i} \binom{m+i-1}{i-1} \\
&= (-)^{i+m+1} \binom{n}{m+i} \left\{ \frac{(i+m)!}{(i-1)!m!} \sum_{k=0}^{m-1} \frac{(-)^k}{k+i} \binom{m}{k} + (-)^m \binom{m+i-1}{i-1} \right\} \\
&= (-)^{i+m+1} \binom{n}{m+i} \left\{ \frac{(i+m)!}{(i-1)!\,m!} \sum_{k=0}^{m} \frac{(-)^k}{k+i} \binom{m}{k} \right\} \\
&\equiv \mathrm{d}g(m)\, \{h(m)\}.
\end{aligned}
$$

We need to show that $h(m) = 1$, which is implied by

$$\sum_{k=0}^{m} \frac{(-)^k}{k+i} \binom{m}{k} = \sum_{k=0}^{m} \int_0^1 \mathrm{d}x\, (-)^k x^{k+i-1} \binom{m}{k} = \int_0^1 \mathrm{d}x\, x^{i-1} \sum_{k=0}^{m} \binom{m}{k} x^k$$
$$= \int_0^1 \mathrm{d}x\, x^{i-1}(1-x)^m = \frac{\Gamma(i)\Gamma(m+1)}{\Gamma(m+i+1)}.$$

This ends the proof.

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